# *FACTSHEET*

This factsheet summarises the main methods, formulae and information required for tackling questions on the topics in this booklet.

## *1 Time value of money*

The accumulated value of 1 (invested at time 0) at time $t$ is $(1+ti)$ if we have simple interest and $(1+i)^t$ if we have compound interest.

The discount factor, $v$, is $\dfrac{1}{1+i}$.

The discount factor for a period of $n$ years using a simple rate of discount $d$ is $1-nd$.

The connection between compound rates of interest and discount is $d = 1 - v = 1-(1+i)^{-1}$.

## *2 Interest rates*

The definition of the force of interest $\delta(t)$ is $\lim\limits_{h\to 0+} \dfrac{A(t,t+h)-1}{h}$.

The accumulation factor from time $t_1$ to time $t_2$ is $A(t_1,t_2) = \exp\left[\int_{t_1}^{t_2} \delta(t)\,dt\right]$.

Correspondingly, the discount factor from time $t_2$ to time $t_1$ is $\exp\left[-\int_{t_1}^{t_2} \delta(t)\,dt\right]$.

The present value at time $a$ of a payment stream of $\rho(t)$ received between time $a$ and time $b$ where the force of interest is $\delta(t)$ is:

$$\int_a^b \rho(t)\exp\left[-\int_a^t \delta(s)\,ds\right]dt$$

The accumulated value at time $b$ of a payment stream of $\rho(t)$ received between time $a$ and time $b$ where the force of interest is $\delta(t)$ is:

$$\int_a^b \rho(t)\exp\left[\int_t^b \delta(s)\ ds\right] dt$$

If the force of interest is a constant, $\delta$, then $A(t_1,t_2)=\exp\left[\delta(t_2-t_1)\right]$, $i=e^{\delta}-1$, $\delta=\ln(1+i)$, $v=e^{-\delta}$ and $1-d=e^{-\delta}$.

The connections between nominal and effective rates of interest are:

$$1+i=\left(1+\frac{i^{(p)}}{p}\right)^p \qquad i^{(p)}=p\left((1+i)^{\frac{1}{p}}-1\right)$$

The connections between nominal and effective rates of discount are:

$$1-d=\left(1-\frac{d^{(p)}}{p}\right)^p \qquad d^{(p)}=p\left(1-(1-d)^{\frac{1}{p}}\right)$$

***3*** *Annuities*

The present value at time 0 of payments of 1 at time 1, 1 at time 2, 1 at time 3 and so on until 1 at time $n$, is given by $a_{\overline{n}|}$. The formula for $a_{\overline{n}|}$ is:

$$a_{\overline{n}|}=\frac{1-v^n}{i}$$

The present value at time 0 of payments of 1 at time 0, 1 at time 1, 1 at time 2 and so on until 1 at time $n-1$, is given by $\ddot{a}_{\overline{n}|}$. The formula for $\ddot{a}_{\overline{n}|}$ is:

$$\ddot{a}_{\overline{n}|}=\frac{1-v^n}{d}$$

The connections between these values are $\ddot{a}_{\overline{n}|}=(1+i)a_{\overline{n}|}$ and $\ddot{a}_{\overline{n}|}=1+a_{\overline{n-1}|}$.

The accumulated value at time $n$ of payments of 1 at time 1, 1 at time 2, 1 at time 3 and so on until 1 at time $n$, is given by $s_{\overline{n}|}$. The formula for $s_{\overline{n}|}$ is:

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i} = (1+i)^n a_{\overline{n}|}$$

The accumulated value at time $n$ of payments of 1 at time 0, 1 at time 1, 1 at time 2 and so on until 1 at time $n-1$, is given by $\ddot{s}_{\overline{n}|}$. The formula for $\ddot{s}_{\overline{n}|}$ is:

$$\ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{d} = (1+i)^n \ddot{a}_{\overline{n}|}$$

The connections between these values are $\ddot{s}_{\overline{n}|} = (1+i)s_{\overline{n}|}$, and $\ddot{s}_{\overline{n}|} + 1 = s_{\overline{n+1}|}$.

The present value at time 0 of payments of 1 *pa* payable continuously for $n$ years is given by $\bar{a}_{\overline{n}|}$. The formula for $\bar{a}_{\overline{n}|}$ is:

$$\bar{a}_{\overline{n}|} = \frac{1-v^n}{\delta}$$

The accumulated value at time $n$ of payments of 1 *pa* payable continuously for $n$ years is given by $\bar{s}_{\overline{n}|}$. The formula for $\bar{s}_{\overline{n}|}$ is:

$$\bar{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{\delta} = (1+i)^n \bar{a}_{\overline{n}|}$$

The present value at time 0 of payments of 1 *pa* payable $p$ thly in arrears for $n$ years is given by $a^{(p)}_{\overline{n}|}$. The amount of each payment is $\frac{1}{p}$. The formula for $a^{(p)}_{\overline{n}|}$ is:

$$a^{(p)}_{\overline{n}|} = \frac{1-v^n}{i^{(p)}} = \frac{i}{i^{(p)}} a_{\overline{n}|}$$

The present value at time 0 of payments of 1 *pa* payable $p$ thly in advance for $n$ years is given by $\ddot{a}_{\overline{n}|}^{(p)}$. The amount of each payment is $\frac{1}{p}$. The formula for $\ddot{a}_{\overline{n}|}^{(p)}$ is:

$$\ddot{a}_{\overline{n}|}^{(p)} = \frac{1-v^n}{d^{(p)}} = \frac{i}{d^{(p)}} a_{\overline{n}|}$$

The connection between these values is $a_{\overline{n}|}^{(p)} = v^{\frac{1}{p}} \ddot{a}_{\overline{n}|}^{(p)}$.

There are equivalent formulae for $s_{\overline{n}|}^{(p)}$ and $\ddot{s}_{\overline{n}|}^{(p)}$, namely:

$$s_{\overline{n}|}^{(p)} = \frac{(1+i)^n - 1}{i^{(p)}} = (1+i)^n \frac{i}{i^{(p)}} a_{\overline{n}|}, \quad \ddot{s}_{\overline{n}|}^{(p)} = \frac{(1+i)^n - 1}{d^{(p)}} = (1+i)^n \frac{i}{d^{(p)}} a_{\overline{n}|}$$

**4** *Deferred and increasing annuities*

The present value at time 0 of payments of 1 at time $m+1$, 1 at time $m+2$, 1 at time $m+3$ and so on until 1 at time $m+n$, is given by ${}_{m|}a_{\overline{n}|}$. The formula for ${}_{m|}a_{\overline{n}|}$ is:

$${}_{m|}a_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|} = v^m a_{\overline{n}|}$$

The corresponding annuity due, continuously payable annuity and $p$ thly annuities are given by:

$${}_{m|}\ddot{a}_{\overline{n}|} = \ddot{a}_{\overline{m+n}|} - \ddot{a}_{\overline{m}|} = v^m \ddot{a}_{\overline{n}|} \qquad {}_{m|}\bar{a}_{\overline{n}|} = \bar{a}_{\overline{m+n}|} - \bar{a}_{\overline{m}|} = v^m \bar{a}_{\overline{n}|}$$

$${}_{m|}a_{\overline{n}|}^{(p)} = a_{\overline{m+n}|}^{(p)} - a_{\overline{m}|}^{(p)} = v^m a_{\overline{n}|}^{(p)} \qquad {}_{m|}\ddot{a}_{\overline{n}|}^{(p)} = \ddot{a}_{\overline{m+n}|}^{(p)} - \ddot{a}_{\overline{m}|}^{(p)} = v^m \ddot{a}_{\overline{n}|}^{(p)}$$

The present value at time 0 of payments of 1 at time 1, 2 at time 2, 3 at time 3 and so on until $n$ at time $n$, is given by $(Ia)_{\overline{n}|}$. The formula for $(Ia)_{\overline{n}|}$ is:

$$(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}$$

The present value at time 0 of payments of 1 at time 0, 2 at time 1, 3 at time 2 and so on until $n$ at time $n-1$, is given by $(I\ddot{a})_{\overline{n}|}$. The formula for $(I\ddot{a})_{\overline{n}|}$ is:

$$(I\ddot{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d}$$

The present value at time 0 of payments of $r$ made continuously through year $r$ for $n$ years is given by $(I\bar{a})_{\overline{n}|}$. The formula for $(I\bar{a})_{\overline{n}|}$ is:

$$(I\bar{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{\delta}$$

The present value at time 0 of a rate of payment of $t$ at time $t$ for $n$ years is given by $(\bar{I}\,\bar{a})_{\overline{n}|}$. The formula for $(\bar{I}\,\bar{a})_{\overline{n}|}$ is:

$$(\bar{I}\,\bar{a})_{\overline{n}|} = \frac{\bar{a}_{\overline{n}|} - nv^n}{\delta}$$

The accumulated values can be found by accumulating the present values, for example:

$$(Is)_{\overline{n}|} = (1+i)^n (Ia)_{\overline{n}|}$$

Deferred annuities can be calculated in the obvious way, for example:

$${}_{m|}(Ia)_{\overline{n}|} = v^m (Ia)_{\overline{n}|}$$